SECOND ORDER CONCENTRATION NEAR THE BINDING ENERGY OF THE HELIUM SCHRODINGER OPERATOR

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ABSTRACT

Let the unperturbed operator be the helium-Schrodinger operator and let the perturbation be the homogeneous-electric-field operator. It is shown that the first two formal perturbation equations corresponding to the smallest unperturbed eigenvalue do admit solutions in the appropriate Hilbert space. According to general considerations this implies that for this perturbation problem, the phenomenon of spectral concentration holds.

1. **Introduction.** In his third communication on the perturbation of spectra Schrodinger [1] computed the energy levels of the hydrogen atom in a weak electric field. His results coincided with the experiments with great accuracy. Then Oppenheimer pointed out [2] that according to physical intuition the entities computed by Schrodinger cannot be point-eigenvalues of the corresponding Schrodinger operator. Later this fact was rigorously established by Titchmarsh [6].

It was observed by Riddell [15] [14.k] and elsewhere [12] that the phenomenon of spectral concentration holds for a large class of abstract operators, including the family of Schrodinger operators corresponding to hydrogen in an electric held.

It is the purpose of this paper to show that the phenomenon of spectral concentration also holds, near the binding enery of the helium Schrodinger operator. According to a verbal communication of Galindo, the spectra of the Schrodinger operators corresponding to helium in an electric field are unbounded from below. Aside from this there is little known about these spectra. It is possible, although not likely that near the helium binding energy these spectra consist of point-eigenvalues. Then our theorem, to be stated, would be vacuous. Even in

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this case, however, it seems easier to establish the concentration phenomenon than to determine these spectra exactly.

Specifically in §2 we summarize some known facts about the helium Schrodinger operator. Then we apply the abstract concentration theorem of [12] to the family of Schrodinger operators corresponding to helium in a weak homogeneous electric field. Here, of course, the parameter of the family is the strength of the electric field, and we consider the spectra near the helium binding energy. This application is carried out in Theorem 2.1. The key assumption of this theorem is that the first two formal perturbation equations do admit solutions. This assumption is verified in the two sections that follow.

In §3 we consider an abstract perturbation problem and in Lemma 3.1 we formulate conditions which ensure that the formal perturbation equations do admit solutions. These conditions are as one expects them to be and are formulated for convenience. In Lemma 3.2 we introduce further conditions, which imply the main ones of Lemma 3.1.

In §4 we return to the key assumption of Theorem 2.1. In Theorem 4.1 we isolate a property of our perturbation problem. This property is important for us, inasmuch as it will allow us to verify the key assumption of Theorem 2.1. We derive Theorem 4.1 from four lemmas. The statement of Lemma 4.1 holds for a class of elliptic partial differential operators although we formulate it for the Laplacian only. The proof of this lemma is based on the usual technique of mollifying operators [10]. In Lemma 4.2 we observe that a certain ordinary differential operator is accretive and this is our basic lemma. Its assumptions are rather restrictive and this is the reason why we can establish only second order concentration and only near the helium binding energy. For heavier atoms additional point-eigenvalues can be included, however, we shall not be concerned with this fact. Lemmas 4.3 and 4.4 are again general and hold for any isolated point-eigenvalue of the helium-Schrodinger operator. These four lemmas together establish Theorem 4.1. We apply Theorem 4.1 in conjuction with the abstract Lemmas 3.1 and 3.2 to our perturbation problem. Lemma 3.2 allows us, so to speak, to isolate the effect of each of the electron-nucleus potentials in the ground states of the helium-Schrodinger operator. This intuitive idea is behind the two rigorous applications of Lemma 3.2 at the end of Section 4.

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2. **Formulation of the Concentration Theorem.** For the unperturbed operator we take the helium-Schrodinger operator. To describe it in more specific terms let $\dot{\mathfrak{C}}_\infty(\mathscr{E}_3)$ denote the class of infinitely differentiable functions with bounded support in $\mathscr{E}_3$, the real Euclidean space of dimension 3. The Schrodinger operator in atomic units [5.e], corresponding to the helium ion He^+, is given on $\dot{\mathfrak{C}}_\infty(\mathscr{E}_3)$ by

$$(\dot{\text{He}}^{+})f = -\frac{1}{2}\Delta f - 2M\left(\frac{1}{r}\right)f, \qquad f \in \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_3). \tag{2.1}$$

Here and in the following we use a dot to emphasize that a given operator is defined on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_3)$. In Equation (2.1) Δ denotes the Laplacian and $M(1/r)$ denotes the operator of multiplication by the function

$$\frac{1}{r}(x) = \frac{1}{|x|}, \qquad x \in \mathscr{E}_3.$$

According to Kato [3], [14.i] the operator $\dot{\text{He}}^{+}$ is essentially self-adjoint on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_3)$ and the domain of its closure equals the domain of the closure of $\dot{\Delta}$. That is

$$\mathfrak{D}(\text{He}^{+}) = \mathfrak{D}(\Delta).$$

Next define the function q on $\mathscr{E}_6$ by

$$q(x) = ((x_6 - x_3)^2 + (x_5 - x_2)^2 + (x_4 - x_1)^2)^{-1/2} \tag{2.2}$$

and let $M(\dot{q})$ on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ be the operator of multiplication by this function. Using the usual notations for the Kroneker product of operators [18], the helium-Schrodinger operator in atomic units is given on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ by [5.b]

$$\dot{\text{He}} = \dot{\text{He}}^{+} \otimes I + I \otimes \dot{\text{He}}^{+} + \dot{M}(q), \tag{2.3}$$

where I denotes the identity operator on $\mathfrak{L}_2(\mathscr{E}_3)$. Actually, it would be sufficient to define this operator on

$$\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_3) \dot{\otimes} \mathfrak{C}_{\infty}(\mathscr{E}_3) \subset \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6),$$

but we shall not be concerned with this fact. Remembering definition (2.1) we see that

$$\dot{\text{He}} = -\frac{1}{2}[\dot{\Delta} \otimes I + I \otimes \dot{\Delta}] + \left[-2M\left(\dot{\frac{1}{r}}\right) \otimes I - 2I \otimes M\left(\dot{\frac{1}{r}}\right)\right] + M(\dot{q}).$$

According to Kato [3] [14.1] this operator is essentially selfadjoint on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ and the domain of its closure equals the domain of the Laplacian. More specifically for the domains of the closures of these operators we have the following inclusions,

$$\mathfrak{D}(\text{He}) = \mathfrak{D}(\Delta \otimes I + I \otimes \Delta) \tag{2.4}$$

$$\mathfrak{D}(\text{He}) \subset \mathfrak{D}\left(M\left(\frac{1}{r}\right) \otimes I\right), \quad \mathfrak{D}(\text{He}) \subset \mathfrak{D}\left(I \otimes M\left(\frac{1}{r}\right)\right) \tag*{(2.5)$^{(1)(2)}$}$$

$$\mathfrak{D}(\mathrm{He}) \subset \mathfrak{D}(M(q)). \tag{2.6}$$

For brevity, we shall set

$$M^{(1)}\left(\frac{1}{r}\right) = M\left(\frac{1}{r}\right) \otimes I$$

$$M^{(2)}\left(\frac{1}{r}\right) = I \otimes M\left(\frac{1}{r}\right)$$

and

$$\Delta^{(1,2)} = \Delta \otimes I + I \otimes \Delta$$

Note that superscripts correspond to the decomposition

$$\mathfrak{L}_2(\mathscr{E}_6) = \mathfrak{L}_2(\mathscr{E}_3) \otimes \mathfrak{L}_2(\mathscr{E}_3).$$

For the perturbation V we take the operator corresponding to a homogeneous electric field. More specifically it is the closure of the operator $\dot{V}$ given by

$$\dot{V}f(x) = x_3 + x_6)f(x), \; x \in \mathscr{E}_6, \; f \in \mathfrak{C}_\infty(\mathscr{E}_6). \tag{2.7}$$

We define the family of perturbed operators on $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ by setting

$$\dot{H}(\varepsilon) = \dot{\mathrm{H}}\mathrm{e} + \varepsilon \dot{V}. \tag{2.8}$$

More specifically, we denote by $H(\varepsilon)$ an arbitrary self-adjoint extension of this operator. Since the operators $\dot{H}(\varepsilon)$ commute with conjugation, the existence of such an extension is ensured by a theorem of von Neumann [8]. We do not know whether such an extension is unique or not. The corresponding question for the hydrogen Schrodinger operator was treated by Ikebe and Kato [9]. They showed that in this case the extension is unique. Note that at least formally, the operator $H(\varepsilon)$ is the Schrodinger operator corresponding to the helium atom in a homogeneous electric field of intensity ε [5.d].

Next let $H(\varepsilon)$ be a given family of self-adjoint operators acting in an abstract Hilbert space. For a given Borel subset $\mathscr{B}_\varepsilon$ of the real line let $E(\varepsilon, \mathscr{B}_\varepsilon)$ denote the spectral projector of $H(\varepsilon)$ over $\mathscr{C}_\varepsilon$. Following a terminology used elsewhere [12], we shall say that near a given point λ_0, the spectrum of the family of operators $H(\varepsilon)$ is concentrated to order p, if there is a family of sets $\mathscr{B}_\varepsilon$, such that

$$E(\varepsilon, \mathscr{B}_\varepsilon) \to E(0, \{\lambda_0\}) \;\; as \;\; \varepsilon \to 0,$$

and

$$\left| \mathscr{B}_\varepsilon \right| = o(\varepsilon^p) \;\; at \;\; \varepsilon = 0.$$

Here, the left member, denotes the Lebesgue measure of $\mathscr{B}_\varepsilon$ and convergence means strong convergence.

After these preparations we return to the family of Schrodinger operators in (2.8). As mentioned in the introduction, in this paper we show that near the binding energy of the helium-Schrodinger operator, the spectrum of these operators is concentrated in this technical sense. This is the statement of the theorem that follows. In it for a given operator T we denote by $\sigma(T)$ its spectrum.

THEOREM 2.1. *Let the helium Schrodinger operator He be defined by equation* (2.3) *and let the family of operators* $H(\varepsilon)$ *be defined by equation* (2.8). *Set,*

$$\lambda_b = \inf \sigma(\mathrm{He}). \tag{2.9}$$

Then near the point λ_b *the spectra of the family of operators* $H(\varepsilon)$ *is concentrated to order two.*

According to a theorem obtained by Riddell [15] [14.k] and elsewhere [12], the phenomenon of spectral concentration obtains under general circumstances. To describe these circumstances, following Kato [14.c], we say that a given subset $\mathfrak{S}$ of $\mathfrak{D}(T)$, is a core of the given operator T if the closure of its restriction to $\mathfrak{S}$ equals T. In particular, if T is essentially self-adjoint on $\mathfrak{S}$ then $\mathfrak{S}$ is a core of T., Using this notion the assumptions of the abstract spectral concentration theorem adapted to our operators, can be stated as follows:

(2.10) λ_b *is an isolated point-eigenvalue of* He *of finite m-multiplicity,*

(2.11) *as* ε *converges to zero, the operators* $H(\varepsilon)$ *converge strongly to* He *on a set which is a core of the unperturbed operator* He,

(2.12) *the first two formal perturbation equations corresponding to the family* $H(\varepsilon)$ *at the point* λ_b *admit m-linearly independent solutions.*

Thus to establish Theorem 2.1 it suffices to establish these three conditions.

To verify condition (2.10) we need a theorem of Zhiclin [7] and Jorgens [20]. This says that the essential spectrum of the helium Schrodinger operator is given by

$$\sigma_e(\mathrm{He}) = [\inf \sigma(\mathrm{He}^+), \infty).$$

As is well known [5.a],

$$\inf \sigma(\mathrm{He}^+) = -2. \tag{2.13}$$

It is also known [5.c] that the value $-\left(2 - \frac{5}{16}\right)$ is an upper bound of λ_b. Hence

$$\lambda_b \leqq -\left(2 - \frac{5}{16}\right)^2 = -\left(\frac{27}{16}\right)^2 < -\frac{13.14}{8^2} < -2 - \frac{3}{4} \tag{2.14}$$

and we see that λ_b is not in the essential spectrum of the operator He. That is to say, condition (2.10) holds for the value λ_b.

To verify condition (2.11) recall that according to Kato the operator He is essentially self-adjoint on $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$. It is clear from definition (2.8) that for each vector f in this set,

$$\lim_{\varepsilon=0} H(\varepsilon)f = \mathrm{He}f.$$

In other words, condition (2.11) holds for this family of operators.

It remains to verify condition (2.12), which we shall do in the two sections that follow.

3. **A sufficient condition for the solvability of the perturbation equations.** Let H and V be possibly unbounded symmetric operators acting in some abstract Hilbert space $\mathfrak{H}$ and assume that the intersection of their domains is dense. Define the family of operators $H(\varepsilon)$ by

$$H(\varepsilon) = H + \varepsilon V \ \ on \ \mathfrak{D}(H) \cap \mathfrak{D}(V). \tag{3.1}$$

Suppose that λ_0 is an isolated point eigenvalue of H and formally set

$$\lambda(\varepsilon) \sim \sum_{j=0}^{\infty} \lambda_j \varepsilon^j \tag{3.2}$$

and

$$f(\varepsilon) \sim \sum_{j=0}^{\infty} f_j \varepsilon^j \tag{3.3}$$

and

$$H(\varepsilon)f(\varepsilon) \sim \lambda(\varepsilon)f(\varepsilon). \tag{3.4}$$

Carrying out the multiplication of the formal power series in this relation and equating the coefficients of the like powers of ε, we obtain the following set of recursive equations,

$$(H - \lambda_0)f_0 = 0 \tag{3.5_o}$$

$$(H - \lambda_0)f_n = \sum_{j=1}^{n} \lambda_j f_{n-j} - V f_{n-1}, \qquad n = 1, 2, \cdots. \tag{3.5_n}$$

This set of equations is called the set of formal perturbation equations corresponding to the family $H(\varepsilon)$ at the point λ_0. Note that in general $f(\varepsilon)$ is not an eigenvector and $\lambda(\varepsilon)$ is not eigenvalue of $H(\varepsilon)$, in fact such formal power series need not exist. In the lemma that follows we formulate a sufficient condition for the solvability of these equations. These conditions were observed elsewhere [12] and they imply the recent ones of Riddell [15].

LEMMA 3.1. *Let λ_0 be an isolated point eigenvalue of H of finite multiplicity and let $E\{\lambda_0\}$ denote the eigen-projector over λ_0. Suppose that there is a sequence of nested linear sets $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \cdots \subset \mathfrak{S}_n = \mathfrak{H}$, such that*

$$E\{\lambda_0\}\mathfrak{H} \subset \mathfrak{S}_0 \tag{3.6}$$

and for $k = 0, 1, \cdots, n-1$

$$(3.7)_k \qquad V\mathfrak{S}_k \subset \mathfrak{S}_{k+1},$$

and for $k = 0, 1, \cdots, n$

$$(3.8)_k \qquad (\lambda_0 - H + E\{\lambda_0\})^{-1}\mathfrak{S}_k \subset \mathfrak{S}_k.$$

Then the first n formal perturbation equations corresponding to the family (3.1) *at* λ_0, *that is equations* $(3.5)_0$ *through* $(3.5)_n$, *do admit solutions. Furthermore, the number of linearly independent solutions equals* $\dim E\{\lambda_0\}\mathfrak{H}$.

In case the perturbation V is bounded, we set

$$\mathfrak{S}_k = \mathfrak{H}, \qquad k = 0, 1, 2, \cdots.$$

Then clearly assumptions (3.6), $(3.7)_k$ and $(3.8)_k$ hold with reference to this sequence of sets. Even in this case the statement of the lemma is not evident, and for a proof we refer to the book of Friedrichs [11]. Next consider the case of an unbounded perturbation V. Suppose that for some value of $k - 1 \leqq n - 1$ the sequence of numbers $\lambda_0, \lambda_1, \cdots, \lambda_{k-1}$ and the sequence of vectors $f_0, \cdots, f_{k-1}$, satisfy equations $(3.5)_0$ through $(3.5)_{k-1}$. Supose further that

$$f_j \in \mathfrak{S}_j \subset \mathfrak{D}(V) \qquad j = 0, 1, \cdots, k-1$$

and that for a given λ_k equation $(3.5)_k$ does admit solutions. Then we see from assumptions (3.6), $(3.7)_{k-1}$ and $(3.8)_k$ that these solutions are given by

$$(3.9)_k \qquad f_k = -(\lambda_0 - H + E\{\lambda_0\})^{-1}\left(\sum_{j=1}^{k} \lambda_j f_{k-j} - Vf_{k-1}\right) + g_k,$$

where g_k is an arbitrary vector in $E\{\lambda_0\}\mathfrak{H}$. At the same time we see that,

$$f_k \subset \mathfrak{S}_k.$$

If also $k \leqq n - 1$, then we see from assumption $(3.7)_k$ that f_k is in $\mathfrak{D}(V)$. This fact allows us to replace k by $k + 1$ in formula $(3.9)_k$, provided that the vectors $g_0, \cdots g_k$, and the numbers $\lambda_0, \cdots \lambda_k$ are chosen appropriately. Now the problem of this appropriate choice is the same for bounded and unbounded perturbations. For brevity we do not repeat this argument [11] and consider the proof of Lemma 3.1 complete.

The assumptions of Lemma 3.1 are rather general and it is difficult to verify them for specific operators. In the lemma that follows we formulate assumptions, which are adapted to our perturbation problem, and which imply the main ones of Lemma 3.1. In it $\mathfrak{B}(\mathfrak{H})$ denotes the space of bounded operators defined on all of $\mathfrak{H}$ and $\rho(T)$ denotes the resolvent set of a given operator T.

LEMMA 3.2. *Let the operators* $A_{0,1}$ *be self-adjoint on the given domains*

$\mathfrak{D}(A_{0,1})$ in $\mathfrak{H}$ and let $\lambda_0 \in \rho(A_0)$ be an isolated eigenvalue of $A_0 + A_1$ of finite multiplicity, with eigenprojector $E\{\lambda_0\}$. Suppose that

$$A_1(\lambda_0 - A_0)^{-1} \in \mathfrak{B}(\mathfrak{H}). \tag{3.10}$$

Suppose further that a sequence of sets $\mathfrak{B}_0 = \mathfrak{H}, \mathfrak{B}_1, \cdots \mathfrak{B}_h$ is given, such that for each integer $l \leqq n-1$,

$$A_1(\mathfrak{B}_l \cap \mathfrak{D}(A_1)) \subset \mathfrak{B}_{l+1} \tag*{(3.11)$_l$}$$

and for each integer $l \leqq n$

$$(\lambda_0 - A_0)^{-1}\, \mathfrak{B}_l \subset \mathfrak{B}_l. \tag*{(3.12)$_l$}$$

Then the sequence of sets

$$\mathfrak{S}_k = \bigcap_{l=0}^{n-k} \mathfrak{B}_l, \qquad k = 0, 1, \cdots n, \tag*{(3.13)$_k$}$$

satisfies assumptions (3.6) *and* $(3.8)_k$ *with reference to the operator $H = A_0 + A_1$.*

To verify assumption (3.6) let h be an arbitrary vector in the range of the eigenprojector $E\{\lambda_0\}$, i.e. set

$$(A_0 + A_1)h = \lambda_0 h. \tag{3.14}$$

We see from assumption (3.10) that

$$\mathfrak{D}(A_0) \subset \mathfrak{D}(A_1),$$

and hence

$$h \in \mathfrak{D}(A_1). \tag{3.15}$$

Insertion of this fact in equation (3.14) yields

$$h = (\lambda_0 - A_0)^{-1} A_1 h. \tag{3.16}$$

Assumption $(3.11)_0$ together with relation (3.15) shows that

$$A_1 h \in \mathfrak{B}_1.$$

Insertion of this inclusion and of assumption $(3.12)_l$ in equation (3.16) yields

$$h \in \mathfrak{B}_1.$$

If $2 \leqq n$, this argument can be repeated and we see that h is also in $\mathfrak{B}_2$. Similarly, we see that

$$h \in \mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_n,$$

and remembering definition $(3.13)_0$, we obtain the validity of assumption (3.6).

To verify assumption (3.8) recall that by assumption the projector $E\{\lambda_0\}$ is of finite rank and hence defined on all of $\mathfrak{H}$. This yields

$$\mathfrak{D}(A_0 + A_1 - E\{\lambda_0\}) = \mathfrak{D}(A_0 + A_1),$$

and in view of assumption (3.10) we have

$$\mathfrak{D}(A_0 + A_1 - E\{\lambda_0\}) = \mathfrak{D}(A_0). \tag{3.17}$$

Since $A_0 + A_1$ is symmetric on $\mathfrak{D}(A_0)$, thre assumption that λ_0 is an isolated eigenvalue implies [14.1] that

$$(\lambda_0 - A_0 - A_1 + E\{\lambda_0\})^{-1} \in \mathfrak{B}(\mathfrak{H}). \tag{3.18}$$

According to general considerations [13] relations (3.17) and (3.18) imply that

$$[1 - (A_1 + E\{\lambda_0\})(\lambda_0 - A_0)^{-1}]^{-1} \in \mathfrak{B}(\mathfrak{H})$$

and

$$(\lambda_0 - A_0 - A_1 + E\{\lambda_0\})^{-1} = (\lambda_0 - A_0)^{-1}\,[1 - (A_1 + E\{\lambda_0\})(\lambda_0 - A_0)^{-1}]^{-1}. \tag{3.19}$$

We maintain that for each $k \leqq n$,

$$[1 - (A_1 + E\{\lambda_0\})(\lambda_0 - A_0)^{-1}]^{-1}\mathfrak{S}_k \subset \mathfrak{S}_k. \tag{3.20$_k$}$$

To verify this let f be an arbitrary vector in $\mathfrak{S}_k$, that is let

$$f \in \mathfrak{B}_0 \cap \mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_{n-k},$$

and set

$$[1 - (A_1 + E\{\lambda_0\})(\lambda_0 - A_0)^{-1}]^{-1} f = g. \tag{3.21}$$

Then clearly

$$g = f + (A_1 + E\{\lambda_0\})(\lambda_0 - A_0)^{-1} g. \tag{3.22}$$

According to the already established assumption (3.6) for each vector g in $\mathfrak{H}$

$$E\{\lambda_0\}(\lambda_0 - A_0)^{-1} g \in \mathfrak{S}_0 \subset \mathfrak{S}_k.$$

Assumptions $(3.11)_o$ and $(3.22)_l$ show that

$$A_1(\lambda_0 - A_0)^{-1} g \in \mathfrak{B}_1.$$

Inserting these two inclusions in equation (3.22) we obtain that g is in $\mathfrak{B}_1$. If $2 \leqq n - k$ then f is in $\mathfrak{B}_2$ and a repetition of this argument shows that also g is in $\mathfrak{B}_2$. Similarly, we see that

$$g \in \mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_{n-k}.$$

Remembering definitions (3.21) and $(3.13)_k$ this establishes the validity of inclusion

$(3.20)_k$. Insertion of this inclusion in equation (3.19) yields the validity of assumption $(3.8)_k$ of Lemma 3.1, if we use assumption $(3.12)_b$ of Lemma 3.2. Since this holds for each $k \leqq n$ the proof of Lemma 3.2 is complete.

In conclusion, let us remark that for specific perturbation problems symmetry arguments give some information on the numerical values of these constants. We shall not be concerned with this fact, merely refer to the book of Wigner [16].

4. **Application of Lemmas 3.1 and 3.2 to the Helium Schrodinger Operator.** We have seen in §2 that in order to establish Theorem 2.1 it suffices to establish condition (2.12), which we shall do in this section.

We start by defining two different splittings of the helium Schrodinger operator and applying the abstract Lemma 3.2 to each of these two splittings. To describe these splittings we introduce two operators by setting

$$(4.1)^{(1)} \qquad \dot{F}^{(1)} = \text{He} - M^{(1)}\left(\frac{1}{r}\right) = -\frac{1}{2}\Delta \otimes I + I \otimes \text{He}^{+} + M(q)$$

$$(4.1)^{(2)} \qquad \dot{F}^{(2)} = \text{He} - M^{(2)}\left(\frac{1}{r}\right) = \text{He}^{+} \otimes I - \frac{1}{2} I \otimes \Delta + M(q)$$

on $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ and define $F^{(1),(2)}$ to be their closures. It is also convenient to introduce two more operators, by setting

$$(4.2) \qquad w(\xi) = \left(\frac{1}{1+\xi^2}\right), \qquad \xi \in \mathscr{E}_1$$

and

$$(4.3)_{3,6} \qquad M_{3,6}(w)f(x) = w(x_{3,6})f(x), \qquad x \in \mathscr{E}_6, \quad f \in \dot{\mathfrak{C}}_\infty(\mathscr{E}_6).$$

Note that the subscripts refer to the decomposition,

$$\mathfrak{L}_2(\mathscr{E}_6) = \mathfrak{L}_2(\mathscr{E}_1) \otimes \mathfrak{L}_2(\mathscr{E}_1) \otimes \mathfrak{L}_2(\mathscr{E}_1) \otimes \mathfrak{L}_2(\mathscr{E}_1) \otimes \mathfrak{L}_2(\mathscr{E}_1) \otimes \mathfrak{L}_2(\mathscr{E}_1).$$

We shall make essential use of a property of these operators, which is formulated in the theorem that follows. In it for a given operator T we denote by $\mathfrak{R}(T)$ its range.

THEOREM 4.1. *Let the operators $F^{(1),(2)}$ be defined by equations $(4.1)^{(1),(2)}$. Suppose that the real numbers λ and μ are such that μ is positive and*

$$(4.4) \qquad \lambda < \inf \sigma(\text{He}^{+}) - \frac{\mu^2}{2}.$$

Then

$$(4.5)^{(1),(2)} \qquad \mathfrak{R}\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda - F^{(1),(2)})M_{3,6}(w^{\mu})\right) = \mathfrak{L}_2(\mathscr{E}_6).$$

The proof of this theorem is the main part of this section. We shall derive it from the four lemmas that follow.

LEMMA 4.1. *Let the operators* $F^{(1),(2)}$ *be defined by equations* $(4.1)^{(1),(2)}$ *and let the operators* $M_{3,6}(w^{\mu})$ *be defined by equations* $(4.3)_{3,6}$. *Then for each positive number* μ, *the set* $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ *is a core for the operators*

$$M_{3,6}\left(\frac{1}{w}\right)^{\mu}F^{(1),(2)}M_{3,6}(w^{\mu}).$$

For brevity we shall establish Lemma 4.1 for one of the operators only. The proof will employ the technique of mollifying operators. In fact, we start the proof with a general proposition concerning such operators. In it, let j be a given non-negative function in $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ such that

$$\int j(u)du = 1,$$

and define the sequence of integral operators J_n by the kernel

$$J_n(x,y) = n^6 j(n(x-y)). \tag{4.6}_n$$

PROPOSITION 4.1. *For each integer n let the operator* J_n *be defined by the kernel in* $(4.6)_n$ *and let the function w be defined by equation* (4.2). *Then for each positive* μ,

$$M_3\left(\frac{1}{w}\right)^{\mu}\Delta^{(1,2)}J_nM_3(w^{\mu})\in\mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)) \tag{4.7}_n$$

and for each vector g in $\mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^{\mu}\right)$,

$$\lim_{n=\infty} M_3\left(\frac{1}{w}\right)^{\mu}J_n g = M_3\left(\frac{1}{w}\right)^{\mu}g. \tag{4.8}$$

To verify conclusion $(4.7)_n$ first note that

$$M_3(w^{\mu})\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6) = \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6) \text{ and } J_n\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)\subset\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6). \tag{4.9}_n$$

Hence the operator in $(4.7)_n$ is defined on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$. We maintain that its restriction to this set is bounded. That is

$$\left\|\left(M_3\left(\frac{1}{w}\right)^{\mu}\Delta^{(1,2)}J_nM_3(w^{\mu})\right)\right\| < \infty. \tag{4.10}_{n,\mu}$$

For, an elementary application of the Green's formula shows that the kerne of this operator is given by

$$n^2(\Delta j(n(x-y)))\frac{w^{\mu}(y_3)}{w_{\mu}(x_3)}.$$

Since by assumption the function j is in $\mathring{\mathfrak{C}}_\infty(\mathscr{E}_6)$ so is the function (Δj). Hence for frozen n, the support of $(\Delta j)\,(n(x-y))$ is in some neighborhood of the diagonal $x = y$ of $\mathscr{E}_6 \times \mathscr{E}_6$. Specifically there is a positive number $\gamma(n)$ such that for every x, y in $\mathscr{E}_6 \times \mathscr{E}_6$,

$$|x-y| > \gamma(n) \textit{ implies } (\Delta j)(n(x-y)) = 0.$$

The triangle inequality shows that

$$|x_3| < |x-y| + |y_3|, \text{ hence } x_3^2 \leqq 2|x-y|^2 + 2y_3^2.$$

This yields

$$\frac{1+x_3^2}{1+y_3^2} < 2|x-y|^2 + 2,$$

and remembering definition (4.2) we see that there is a constant $0(1)$ such that for every x, y in $\mathscr{E}_6 \times \mathscr{E}_6$,

$$|x-y| < \gamma(n) \textit{ implies } \frac{w^\mu(y_2)}{w^\mu(x_3)} = 0(1).$$

These two implications together show that this constant is such that

$$(4.11)_n \qquad (\Delta j)(n(x-y))\,\frac{w^\mu(y_3)}{w^\mu(x_3)} = 0(1)(\Delta j)\,|(n(x-y))|.$$

Hence

$$\sup_x \int |(\Delta j)\,(n(x-y))\frac{w^\mu(y_3)}{w_\mu(x_3)}|\,dy + \sup_y \int |(\Delta j)\,(n(x-y))\;\frac{w^\mu(y_2)}{w_\mu(x_3)}\,dx < \infty.$$

According to a result of Holmgren [14.a] this relation implies the validity $(4.10)_{n,\mu}$ if we remember that the kernel of the operator of $(4.10)_n$ is n^2 times the left member of $(4.11)_n$.

Note that for $\mu = 0$, these arguments give the arguments of Friedrichs [10] showing that

$$(4.10)_{n,o} \qquad \|(\Delta^{(1,2)} J_n)\| < \infty.$$

It is a general operator-theoretic fact that the closure of the product of two operators equals the product of the closures provided that the closure of the second factor is bounded and everywhere defined. Insertion of this fact in $(4.10)_{n,0}$ yields

$$\Delta^{(1,2)}J_n \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)) \textit{ and } \Delta^{(1,2)}J_n M_3(w^\mu) \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)).$$

That is, the closure of the second factor in $(4.10)_{n,\mu}$ is bounded and everywhere defined. Insertion of this fact and of the previous general operator-theoretic fact in $(4.10)_{n,\mu}$ yields the validity of conclusion $(4.17)_n$.

To verify conclusion (4.8) we need a sharper, but well known [10] version of relation $(4.9)_n$. Specifically we need that for each function g in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ the union of the supports of the functions

$$J_n M_3(w^\mu)g, \quad n = 1,2,\cdots,$$

is a bounded subset of $\mathscr{E}_6$. Since J_n converges strongly to the identity operator on $\mathfrak{L}_2(\mathscr{E}_6)$, we obtain

$$\lim_{n=\infty} M_3\left(\frac{1}{w}\right)^\mu J_n M_3(w^\mu)g = g, \ \ g \in \dot{\mathfrak{C}}_\infty(\mathscr{E}_6).$$

A repetition of the arguments leading to conclusion $(4.17)_n$ shows that

$$\sup_n \left\| M_3\left(\frac{1}{w}\right)^\mu J_n M_3(w^\mu) \right\| < \infty .$$

That is to say, this sequence of operators is uniformly bounded and on a dense subset and it converges strongly to the identity operator. Therefore for each vector f in $\mathfrak{L}_2(\mathscr{E}_6)$,

$$\lim_{n=\infty} M_3 \left(\frac{1}{w}\right)^\mu J_n M_3(w^\mu)f = f.$$

Setting

$$g = M_3(w^\mu)f,$$

in this relation we obtain the validity of conclusion (4.8).

In the proof of Lemma 4.1 we shall also need some information on the commutator of $\Delta^{(1,2)}$ and $M_3\left(\frac{1}{w}\right)^\mu$. This commutator is defined by

$$\left[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^\mu\right] = \Delta^{(1,2)} M_3\left(\frac{1}{w}\right)^\mu - M_3\left(\frac{1}{w}\right)^\mu \Delta^{(1,2)} \ \ on \ \dot{\mathfrak{C}}_\infty(\mathscr{E}_6),$$

and by closure elsewhere. The existence of this closure is ensured by the anti-symmetric character of this commutator. To describe this commutator let D_3 be the $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$-closure of the operator

$$\dot{D}_3 f(x) = \frac{\partial}{\partial x_3} f(x), \quad x \in \mathscr{E}_6, \ f \in \dot{\mathfrak{C}}_\infty(\mathscr{E}_6).$$

Similarly, let $D\left(\frac{1}{w}\right)^\mu$ denote the derivative of the function $\left(\frac{1}{w}\right)^\mu$.

PROPOSITION 4.2. *Suppose that the vector h is such that*

$$(4.12)_\mu \qquad h \in \mathfrak{D}(\Delta^{(1,2)}) \cap \mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^\mu\right) \text{ and } \Delta^{(1,2)}h \in \mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^\mu\right).$$

Then h is in the domain of the $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ closure of the operator $M_3\left(D\left(\frac{1}{w}\right)^\mu\right) D_3$, that is

$$(4.13)_\mu \qquad h \in \mathfrak{D}\left(M_3\left(D\left(\frac{1}{w}\right)^\mu\right)D_3\right) .$$

Furthermore

$$(4.14)_\mu \qquad [\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^\mu [h = M_3\left(D\left(\frac{1}{w}\right)^\mu\right)D_3 h + M_3\left(D^2\left(\frac{1}{w}\right)^\mu\right)h.$$

To verify conclusion $(4.13)_\mu$ we first maintain that for each real v and function h in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$,

$$(4.15)_v \quad \sum_{i=1}^{6}\left(M_3\left(\frac{1}{w}\right)^v D_i h, D_i h\right) + \left(M_3\left(D\left(\frac{1}{w}\right)^v\right)h, D_3 h\right) + \left(M_3\left(\frac{1}{w}\right)^v h, \Delta h\right) = 0.$$

For, the product rule of differentiation yields

$$\frac{\partial}{\partial x_i}(1+x_3^2)^v h(x)\frac{\partial h(x)}{\partial x_i} = \frac{\partial(1+x_3^2)^v}{\partial x_i}h(x)\,\frac{\partial h(x)}{\partial x_i}$$
$$+ (1+x_3^2)^v\left|\frac{\partial h(x)}{\partial x_i}\right|^2 + (1+x_3^2)^v h(x)\,\frac{\partial^2 h(x)}{\partial x_i^{\,2}}.$$

Since the left member is a partial derivative of a function in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$, integrating with respect to $dx = dx_1 \cdots dx_6$ and summing over $i = 1, \cdots 6$, we obtain the validity of relation $(4.15)_v$.

Next we derive conclusion $(4.13)_l$ from relation $(4.15)_l$. According to the considerations of Friedrichs [10], $(4.12)_l$ implies the existence of a sequence of vectors h_n in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$, such that

$$\lim_{n=\infty} h_n = h,$$

and

$$\{\Delta h_n\} \text{ and } \left\{M_3\left(\frac{1}{w}\right)h_n\right\} \text{ are Cauchy sequences.}$$

We see from definition (4.2) that

$$\left\|M_3\left(D\left(\frac{1}{w}\right)\right)h\right\| \leqq \left\|M_3\left(\frac{1}{w}\right)h\right\|,$$

and clearly

$$2\|D_3h\| \leqq 2\|h\|^{1/2}\cdot\|\Delta h\|^{1/2} \leqq \|h\| + \|\Delta h\|.$$

Hence

$$\{D_3 h_n\} \textit{ and } \left\{M_3\left(D\left(\frac{1}{w}\right)\right) h_n\right\} \textit{ are also Cauchy sequences.}$$

Insertion of these facts in relation $(4.15)_l$ shows that each term of the sum is a Cauchy sequence. Clearly

$$\left(M_3\left(\frac{1}{w}\right)D_3(h_n - h_m),\ D_3(h_n - h_m)\right) = \left\|M_3\left(\frac{1}{w}\right)^{1/2} D_3(h_n - h_m)\right\|^2,$$

and according to definition (4.2)

$$\left\|M_3\left(D\left(\frac{1}{w}\right)\right)D_3(h_n - h_m)\right\| < 2\left\|M_3\left(\frac{1}{w}\right)^{1/2} D_3(h_n - h_m)\right\|.$$

Combining these facts we arrive at the validity of conclusion $(4.13)_l$. An adaptation of these arguments that we shall not carry out, yields the validity of conclusion $(4.13)_\mu$.

To verify conclusion $(4.14)_l$ first note that according to elementary algebra it holds for h in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$. In the general case consider the previously introduced sequence in $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$, $\{h_n\}$. Since

$$\left\|M_3\left(D^2\left(\frac{1}{w}\right)\right)(h_n - h_m)\right\| \leqq \left\|M_3\left(\frac{1}{w}\right)(h_n - h_m)\right\|,$$

we see that for this sequence the right member of $(4.14)_l$ is a Cauchy sequence. Remembering that we defined the domain of the commutator by closure, this yields the validity of conclusion $(4.14)_l$. Similarly we arrive at conclusion $(4.14)_\mu$. This completes the proof of Proposition 4.2.

Having established these two propositions we return to the proof of Lemma 4.1 The key fact of this proof is the inclusion

$$\mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^\mu F^{(1)} M_3(w^\mu)\right) \subset \mathfrak{D}(\Delta^{(1,2)}), \tag{4.16}$$

that we verify presently. Accordingly assume that

$$f \in \mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^\mu F^{(1)} M_3(w^\mu)\right) \tag{4.17}$$

and we shall show that

$$f \in \mathfrak{D}(\Delta^{1,2)}). \tag{4.18}$$

To verify this relation recall conclusion (4.8) of Proposition 4.1. It allows us to define the the sequence of vectors

$$f_n = M_3\left(\frac{1}{w}\right)^{\mu} J_n M_3(w)^{\mu} f. \tag{4.19$_n$}$$

At the same time it shows that

$$\lim_{n=\infty} f_n = f.$$

Since the operator $\Delta^{(1,2)}$ is closed, relation (4.20) is implied by the fact that for each n

$$f_n \in \mathfrak{D}(\Delta^{(1,2)}) \tag{4.21$_n$}$$

and

$$\{\Delta^{(1,2)} f_n\} \text{ is a Cauchy sequence.} \tag{4.22}$$

To verify relation (4.21) recall conclusion $(4.7)_n$ of Proposition 4.1. This shows that for each n, the vector

$$h_n = J_n M_3(w^{\mu}) f$$

satisfies assumption $(4.12)_{\mu}$ of Proposition 4.2. Hence Propositions 4.1 and 4.2 allow us to define the sequence of vectors

$$\begin{aligned} f_n^* = {} & M_3\left(\frac{1}{w}\right)^{\mu} \Delta^{(1,2)} J_n M_3(w^{\mu}) f \\ & + \left[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^{\mu}\right] J_n M_3(w^{\mu}) f. \end{aligned} \tag{4.19$_n^*$}$$

We claim that for every vector h in $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$,

$$(\Delta^{(1,2)} h, f_n) = (h, f_n^*). \tag{4.23}$$

For, we see from definition $(4.19)_n$ that

$$(\Delta^{(1,2)} h, f_n) = \left(M_3(w^{\mu}) J_n M_3\left(\frac{1}{w}\right)^{\mu} \Delta^{(1,2)} h, f\right),$$

and from definition $(4.19)_n^*$ that

$$(H, f_n^*) = \left(M_3(w^{\mu}) J_n \Delta^{(1,2)} M_3\left(\frac{1}{w}\right)^{\mu} h, f\right) - \left(M_3(w^{\mu}) J_n \left[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^{\mu}\right] h, f\right).$$

Since

$$\Delta^{(1,2)} M_3\left(\frac{1}{w}\right)^{\mu} - \left[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^{\mu}\right] = M_3\left(\frac{1}{w}\right)^{\mu} \Delta^{(1,2)} \text{ on } \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6),$$

these two equations establish the validity of relation (4.23). In other words we have shown that the vector f_n is in the domain of the adjoint of $\Delta^{(1,2)}$ in $\mathfrak{C}^{\cdot}_{\infty}(\mathscr{E}_6)$. Since $\Delta^{(1,2)}$ is essentially self-adjoint on this set [14.h] we obtain the validity of relation (4.21).

To verify relation (4.22) insert the already established relation $(4.21)_n$ in definition $(4.19)_n$. This yieds

$$(4.24)_n \qquad \Delta^{(1,2)}f_n = M_3\left(\frac{1}{w}\right)^{\mu}\Delta^{(1,2)}J_nM_3(w^{\mu})f + \left[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^{\mu}\right]J_nM_3(w^{\mu})f.$$

Definition (4.1)[1] together with relations (2.4) and (2.5)[1] shows that assumption (4.19) implies

$$(4.25) \qquad M_3(w^{\mu})f \in \mathfrak{D}(\Delta^{(1,2)}) \textit{ and } \Delta^{(1,2)}M_3(w^{\mu})f \in \mathfrak{D}(M_3\left(\frac{1}{w}\right)^{\mu}.$$

Insertion of this fact in conclusion (4.8) of Proposition 4.1 yields

$$(4.26) \qquad \lim_{n=\infty} M_3\left(\frac{1}{w}\right)^{\mu}\Delta^{(1,2)}J_nM_3(w^{\mu})f = M_3\left(\frac{1}{w}\right)^{\mu}\Delta^{(1,2)}M_3(w^{\mu})f,$$

if we remember that $\Delta^{(1,2)}$ commutes with J_n. According to (4.25) the vector $M_3(w^{\mu})f$ satisfies assumption $(4.12)_{\mu}$. Hence according to conclusion $(4.13)_{\mu}$ of Proposition 4.2

$$D_3M_3(w^{\mu})f \in \mathfrak{D}\left(M_3\left(D\left(\frac{1}{w}\right)^{\mu}\right)\right).$$

This fact together with conclusion (4.8) of Proposition 4.1 implies that

$$\lim_{n=\infty} M_3\left(D\left(\frac{1}{w}\right)^{\mu}D_3J_nM_3(w^{\mu})f = M_3\left(D\left(\frac{1}{w}\right)^{\mu}\right)D_3M_3(w^{\mu})f,$$

if we remember that D_3 and J_n commute. Similarly, we see that

$$\lim_{n=\infty} M_3\left(D^2\left(\frac{1}{w}\right)^{\mu}\right)J_nM_3(w^{\mu})f = M_3\left(D^2\left(\frac{1}{w}\right)^{\mu}\right)M^3(w^{\mu})f.$$

Insertion of these two relations in conclusion $(4.14)_{\mu}$ shows that

$$[\Delta^{(1,2)}, M_3\left(\frac{1}{w}\right)^{\mu}]\ J_nM_3(w^{\mu})f$$

is a Cauchy sequence. Finally, inserting this relation and (4.26) in $(4.24)_n$ we arrive at the validity of relation (4.22).

As mentioned before, relations (4.21) and (4.22) imply relation (4.18). In other words, we have established the validity of inclusion (4.16).

To derive Lemma 4.1 from this inclusion we note that for positive μ,

$$(4.27) \qquad [\Delta^{(1,2)}, M_3(w^{\mu})] = 2M_3(Dw^{\mu})D_3 + M_3(D^2w^{\mu}) \textit{ on } \mathfrak{D}(\Delta^{(1,2}).$$

This relation together with inclusion (4.16) and definitions (4.1)[1] and (2.1) yields

$$M_3\left(\frac{1}{w}\right)^{\mu} F^{(1)} M_3(w^{\mu}) = \frac{1}{2}\Delta^{(1,2)} M_3\left(\frac{1}{w}\right)^{\mu} M_3(Dw^{\mu})D_3 \tag{4.28}$$

$$- \frac{1}{2}M_3\left(\frac{1}{w}\right)^{\mu} M_3(D^2w^{\mu}) + 2I\otimes M\left(\frac{1}{r}\right) + M(q).$$

Definition (4.2) shows that for positive μ,

$$\left\| M_3\left(\frac{1}{w}\right)^{\mu} M_3(Dw^{\mu}) \right\| + \left\| M_3\left(\frac{1}{w}\right)^{\mu} M_3(D^2w^{\mu}) \right\| < \infty.$$

Inserting this fact in (4.28) we obtain that $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ is a core of this operator if we remember that it is a common core of $\Delta^{(1,2)}$ and D_3 and that the last two terms in (4.28) are $\Delta^{(1,2)}$-bounded. This completes the proof of Lemma 4.1.

The lemma that follows will imply that the sum of the second and third term in equation (4.28) is quasi accretive on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$.

LEMMA 4.2. *Let the function w be defined by equation* (4.2). *Then for each positive number μ*

$$0 \leqq \left\{\mu^2 - \mathrm{Re}\left[M\left(\frac{1}{w}\right)^{\mu}[D^2, M(w^{\mu})]\right]\right\} \quad on\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_1). \tag{4.29}$$

To verify conclusion (4.29), we need the $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_1)$ version of the commutator equation (4.27). This shows that

$$-M\left(\frac{1}{w}\right)^{\mu}[D^2, M(w^{\mu})] = -2M\left(\frac{Dw^{\mu}}{w^{\mu}}\right)D + DM\left(\frac{Dw^{\mu}}{w^{\mu}}\right).$$

Since the operators D and $-D$ are adjoint to each other on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_1)$, on this set

$$\mathrm{Re}\left\{-2M\left(\frac{Dw^{\mu}}{w^{\mu}}\right)D\right\} = -M\left(\frac{Dw^{\mu}}{w^{\mu}}\right)D + DM\left(\frac{Dw^{\mu}}{w^{\mu}}\right).$$

Clearly, the commutator on the right equals,

$$M\left(D\frac{Dw^{\mu}}{w^{\mu}}\right) = M\left(\frac{D^2w^{\mu}}{w^{\mu}}\right) - M\left(\frac{Dw^{\mu}}{w^{\mu}}\right)^2 \quad on\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_1).$$

These three equations together show that

$$\mathrm{Re}\left\{-M\left(\frac{1}{w}\right)^{\mu}[D^2, M(w^{\mu})]\right\} = -M\left(\frac{Dw^{\mu}}{w_{\mu}}\right)^2 \quad on\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_1). \tag{4.30}$$

According to definition (4.2)

$$\frac{Dw^{\mu}}{w^{\mu}}(\xi)=\mu\cdot\frac{2\xi}{1+\xi^2},$$

and hence

$$\sup_{\xi}\left(\frac{Dw^{\mu}}{w^{\mu}}\right)^2(\xi)=\mu^2.$$

Insertion of this fact in equation (4.30) yields the validity of conclusion (4.29). This completes the proof of Lemma 4.2..

Now from these two lemmas we can easily derive that the ranges of the operators in Theorem 4.1 are closed. In the following lemma we formulate this fact for future reference.

LEMMA 4.3. *Suppose that the real numbers λ and μ satisfy assumption* (4.4) *Then the ranges of the operators in* $(4.5)^{(1),(2)}$ *are closed.*

Since

$$[\Delta^{(1,2)},M_3(w^{\mu})]=[D_3^2,M_3(w^{\mu})],$$

we see from equations (4.28), (4.27) and definition (2.1) that

$$M_3\left(\frac{1}{w}\right)^{\mu}(F^{(1)}-\lambda)M_3(w^{\mu})=-\frac{1}{2}\Delta\otimes I-\frac{1}{2}M_3\left(\frac{1}{w}\right)^{\mu}[D_3^2,M_3(w^{\mu})]$$

$$(4.28)_{\lambda}\qquad +I\otimes(\mathrm{He}^{+}-\lambda)+M(q),\ \textit{on}\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6).$$

According to assumption (4.4)

$$\frac{\mu^2}{2}<\inf\sigma(\mathrm{He}^{+}-\lambda).$$

It is a general operator-theoretic fact, that this implies

$$\frac{\mu^2}{2}<I\otimes(\mathrm{He}^{+}-\lambda).$$

Similarly we see from conclusion (4.29) of Lemma 4.2 that

$$0\leqq\mathrm{Re}\left\{-\frac{1}{2}M_3\left(\frac{1}{w}\right)^{\mu}[D_3^2,M_3(w^{\mu})]+\frac{\mu^2}{2}\right\}\ \textit{on}\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6).$$

Inserting these two inequalities in equation $(4.28)_{\lambda}$ we obtain

$$(4.31)\qquad 0\leqq\mathrm{Re}\left(f,M_3\left(\frac{1}{w}\right)^{\mu}(F^{(1)}-\lambda)M_3(w^{\mu})f\right)\ \textit{on}\ \dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$$

if we remember that the operator $M(q)$ is positive and that the order of taking the real part of an operator and its quadratic form can be interchanged.

An operator satisfying relation (4.31) is called accretive on $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$. It is well known [14.$f$] that if the operator T is accretive on any of its cores then for each positive number ε the range of $T+\varepsilon$ is closed. We see from assumption (4.4) that to the number λ there is a positive ε such that $\lambda+\varepsilon$ also satisfies this assumption, i.e.,

$$\lambda + \varepsilon < \inf \sigma(\mathrm{He}^+) - \frac{\mu^2}{2}.$$

Aside from a minus sign, the operator in $(4.5)^{(1)}$ can be written as

$$(4.32)\qquad M_3\left(\frac{1}{w}\right)^{\mu}(F^{(1)}-\lambda)M_3(w^{\mu}) = M_3\left(\frac{1}{w}\right)^{\mu}(F^{(1)}-(\lambda+\varepsilon))M_3(w^{\mu})+\varepsilon.$$

According to Lemma 4.1 $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ is a core of the first term and according to (4.31) this operator is accretive on this set. Since by definition ε is strictly positive these facts together with equation (4.32) establish the validity of Lemma 4.3.

We complete the proof of Theorem 4.1 by showing that, under general circumstances, this range is also dense. This is the statement of the lemma that follows. In it, for a given set $\mathfrak{S}$ we denote by $\bar{\mathfrak{S}}$ its closure.

Lemma 4.4. *Suppose that the real number λ is such that*

$$(4.33)\qquad \lambda < \inf \sigma(\mathrm{He}^+).$$

Then, for each positive number μ,

$$(4.34)^{(1),(2)}\qquad \overline{M_3\left(\frac{1}{w}\right)^{\mu}(\lambda - F^{(1),(2)})M_3(w^{\mu})\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)} = \mathfrak{L}_2(\mathscr{E}_6).$$

For brevity we shall verify only one of the conclusions, say $(4.34)^{(1)}$ Let $\mathscr{B}_r$ denote the ball in $\mathscr{E}_6$ of radius r, i.e. set

$$\mathscr{B}_r = \{x : |x| \leqq r,\ x \in \mathscr{E}_6\}.$$

In analogy with previous notation we define the class $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$ and the space $\mathfrak{L}_2(\mathscr{B}_r)$. For a given operator T we denote by T_r the $\mathfrak{L}_2(\mathscr{B}_r)$ closure of its restriction to $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$. It is implicit in this notation that this closure does exist which happens under general circumstances [14.c].

We maintain that the operator $F_r^{(1)}$ is essentially self-adjoint on $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$. For, according to a result of Kato [14.i], to each positive number ε there is a number $\gamma(\varepsilon)$ such that for every function f in $\dot{\mathscr{E}}_\infty(\mathscr{E}_6)$, in particular in $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$,

$$\left\| \left(-2I \otimes M\left(\frac{1}{r}\right) + M(q)\right) f \right\| \leqq \varepsilon \left\| \frac{1}{2}\Delta^{(1,2)} f \right\| + \gamma(\varepsilon)\|f\|.$$

In other words, the restriction of the operator $-2I\otimes M\left(\frac{1}{r}\right)+M(q)$ to $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$ is is bounded with reference to the restriction of the operator $\frac{1}{2}\Delta^{(1,2}$ to $\dot{\mathfrak{C}}_{\infty}(\mathscr{B}_r)$. As is well known [10] the restriction of $\frac{1}{2}\Delta^{(1,2)}$ to $\dot{\mathfrak{C}}_{\infty}(\mathscr{B}_r)$ is essentially self-adjoint in $\mathfrak{L}_2(\mathscr{B}_r)$. According to a theorem of Rellich and Kato [14.g], these two facts together imply that the sum of these two operators is essentially self-adjoint on $\dot{\mathfrak{C}}_{\infty}(\mathscr{B}_r)$ in $\mathfrak{L}_2(\mathscr{B}_r)$. At the same time it follows [14.g] that in this case, the closure of the sum equals the sum of the closure. Remembering definitions (4.1) $^{(1)}$and (2.1) this establishes the essential self-adjointness of $F_r^{(1)}$ on $\dot{\mathfrak{C}}_{\infty}(\mathfrak{B}_r)$, as we have maintained.

We see from this essential self-adjointness that the numerical range of $F_r^{(1)}$ is contained in the closure of the numerical range of its restriction $\dot{\mathfrak{S}}_{\infty}(\mathscr{B}_r)$. Since $F^{(1)}$ is essentially self-adjoint on $\dot{\mathfrak{C}}_{\infty}(\mathscr{E}_6)$ and $\dot{\mathfrak{S}}_{\infty}(\mathscr{B}_r)$ is a subset of $\dot{\mathfrak{S}}_{\infty}(\mathscr{E}_6)$, we obtain that the closure of the numerical range of $F_r^{(1)}$ is contained in closure of the numerical range of $F_r^{(1)}$. In symbols,

$$\nu(F_r^{(1)} \subset \nu(F^{(1)}). \tag{4.35}$$

It is an elementary consequence of the spectral theorem that the convex hull of the spectrum of a self-adjoint operator is closed and that it equals the closure of the numerical range. Applying this fact to the operator $F^{(1)}$ we obtain,

$$\overline{(F^{(1)})} \subset [\inf \sigma(He^+), \infty).$$

Combining this inclusion with (4.35) yields

$$\overline{\nu(F_r^{(1)})} \subset [\inf \sigma(He^+), \infty).$$

Since $F_r^{(1)}$ is self-adjoint, this in turn, yields

$$\sigma(F_r^{(1)}) \subset [\inf \sigma(He^+), \infty).$$

It is a general operator theoretic fact that

$$\sigma\left(-\frac{1}{2}\Delta \otimes I + I \otimes \mathrm{He}^+ = \inf \sigma(\mathrm{He}^+), \infty\right).$$

This yields

$$\sigma(F^{(1)}) \subset (\inf \sigma(\mathrm{He}^+), \infty),$$

if we remember that $M(q)$ is positive and definition $(4.1)^{(1)}$. Actually in this relation the inclusion sign can be replaced by the equality sign as Zhislin [7] has shown it. However, we shall not use this fact. All that we need is that assumption (4.33) implies that

$$\lambda \in \rho(F^{(1)}) \text{ and } \lambda \in \rho(F_r^{(1)}). \tag{4.36}$$

By definition $\dot{\mathfrak{C}}_{\infty}(\mathscr{B}_r)$ is a core of $F_r^{(1)}$ and clearly the image of the core is dense in the range. Hence for every complex number ζ we have

$$\zeta \in \rho(F_r^{(1)}) \textit{ implies } \overline{(\zeta - F_r^{(1)})\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)} = \mathfrak{L}_2(\mathscr{B}_r)$$

According to relation (4.36) we can set $\zeta = \lambda$ in this implication, and we obtain that for each positive number r,

$$\overline{(\lambda - F^{(1)})\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)} = \mathfrak{L}_2(\mathscr{B}_r),$$

if we remember that on $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$ the operators $F^{(1)}$ and $F_r{}^{(1)}$are equal. It is clear from definition (4.2) that for each positive μ

$$M_3(w^*)\dot{\mathfrak{C}}_\infty(\mathscr{B}_r) = \dot{\mathfrak{C}}_\infty(\mathscr{B}_r).$$

At the same time we see that for a given subset $\mathfrak{S}$ of $\mathfrak{L}_2(\mathscr{B}_r)$,

$$\dot{\mathfrak{S}} = \mathfrak{L}_2(\mathscr{B}_r) \textit{ implies } \overline{M_3\left(\frac{1}{w}\right)^\mu \mathfrak{S}} = \mathfrak{L}_2(\mathscr{B}_r).$$

These three relations together show that for each positive μ and r,

$$\overline{M_3\left(\frac{1}{w}\right)^\mu (\lambda - F^{(1)})M_3(w^\mu)\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)} = \mathfrak{L}_2(\mathscr{B}_r).$$

Remembering that $\dot{\mathfrak{C}}_\infty(\mathscr{B}_r)$ is a subset of $\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)$ we arrive at

$$\overline{M_3\left(\frac{1}{w}\right)^\mu (\lambda - F^{(1)})M_3(w^\mu)\dot{\mathfrak{C}}_\infty(\mathscr{E}_6)} \supset \bigcup_{r=0}^{\infty} \mathfrak{L}_2(\mathscr{B}_r).$$

Since the right member is dense in $\mathfrak{L}_2(\mathscr{E}_6)$ this relation establishes the validity of conclusion $(4.34)^{(1)}$. A similar argument that we shall not carry out yields the validity of conclusion $(4.34)^{(2)}$. This completes the proof of Lemma 4.4. Clearly combining Lemmas 4.4 and 4.3 we obtain the validity of Theorem 4.1.

We shall use this theorem via the following:

COROLLARY 4.1. *Let the function w be defined by equation* (4.2) *and let the operators $F^{(1)(2)}$ be defined by equations* $(4.1)^{(1)(2)}$. *Then the binding energy λ_b of definition* (2.9) *is such that for $\mu = 0, \frac{1}{2}, 1$ we have*

$$(4.38) \qquad (\lambda_b - F^{(1)(2)})^{-1}\mathfrak{D}\left(M_{3,6}\left(\frac{1}{w}\right)^\mu\right) \subset \mathfrak{D}\left(M_{3,6}\left(\frac{1}{w}\right)^\mu\right).$$

To verify conclusion (4.38) recall relations (2.13) and (2.14) which show that for λ_b and for these values of μ assumption (4.4) holds. That is to say

$$(4.39) \qquad \lambda_b < \inf \sigma(\mathrm{He}^+) - \frac{\mu^2}{2}, \textit{ for } \mu = 0, \frac{1}{2}, 1.$$

Hence we see from Theorem 4.1 that for such μ

(4.40) $$\mathfrak{R}\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu} \cdot (\lambda_b - F^{(1)(2)}) \cdot M_{3,6}(w^{\mu})\right) = \mathfrak{L}_2(\mathscr{E}_6).$$

We maintain that this operator is closed. For, we defined the second factor by closure, and hence it is closed. Remembering relations (2.13) and (4.37) we see that

(4.41) $$(\lambda_b - F^{(1)(2)})^{-1} \subset \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)).$$

This fact together with the bondedness, in particular closedness of the operator $M_3(w^{\mu})$ implies [14.b] that the product of the second and third factors in (4.40) is closed. Since for positive μ

$$\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu}\right)^{-1} = M_{3,6}(w^{\mu}) \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)),$$

we obtain that the triple product in (4.40) is also closed, as we have maintained. Clearly this triple product is one to one. Thus relation (4.40) allows us to apply a corollary [14.d] of the closed graph theorem, which yields

(4.42) $$\left[M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})M_{3,6}(w^{\mu})\right]^{-1} \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)).$$

An elementary algebra shows that

$$M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})M_{3,6}(w^{\mu}) \cdot M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda - F^{(1)(2)})^{-1}M_{3,6}(w^{\mu}) = I$$

on $$\mathfrak{R}\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})M_{3,6}(w^{\mu})\right),$$

and

$$M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})^{-1}M_{3,6}(w^{\mu}) \cdot M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})M_{3,6}(w^{\mu}) = I$$

on $$\mathfrak{D}\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})M_{3,6}(w^{\mu})\right).$$

These two equations together with relation (4.42) show that

(4.43) $$M_{3,6}\left(\frac{1}{w}\right)^{\mu}(\lambda_b - F^{(1)(2)})^{-1}M_{3,6}(w^{\mu}) \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)).$$

According to definition (4.2) for each positive μ the function w^{μ} is bounded and hence

$$\mathfrak{D}\left(M_{3,6}\left(\frac{1}{w}\right)^{\mu}\right) = M_{3,6}(w^{\mu})\,\mathfrak{L}_2(\mathscr{E}_6).$$

Inserting this equation in relation (4.43) we arrive at the validity of conclusion (4.38). This completes the proof of Corollary 4.1.

Finally we return to the proof of Theorem 2.1. We have already seen in §2 that it suffices to verify condition (2.12). To verify this condition we first mainain that the assumptions of Lemma 3.2. hold for the pair of operators

$$(4.44)^{(1)} \qquad A_0^{(1)} = F^{(1)} \textit{ and } A_1^{(1)} = -2M^{(1)}\left(\frac{1}{r}\right),$$

with reference to the sets

$$(4.45)^{(1)} \qquad \mathfrak{B}_l^{(1)} = \mathfrak{D}\left(M_3\left(\frac{1}{w}\right)^{l/2}\right), \qquad l = 0,1,2.$$

For, we see from relation (4.41) and from relations (2.4) and $(2.5)^{(1)}$ that

$$M^{(1)}\left(\frac{1}{r}\right)(\lambda_b - F^{(1)})^{-1} \in \mathfrak{B}(\mathfrak{L}_2(\mathscr{E}_6)).$$

That is to say assumption (3.10) holds at the point λ_b. The validity of assumption $(3.12)_l$, $l = 0,1,2$, is the statement of Corollary 4.1. To see the validity of assumptions $(3.11)_l$, assume that

$$(4.46) \qquad f \in \mathfrak{D}\left(M^{(1)}\left(\frac{1}{r}\right) \cap \mathfrak{D}(M_3\left(\frac{1}{w}\right)^{l/2}\right).$$

Then it can be written in the form

$$f(x) = \left(\frac{1}{1+x_3^2}\right)^{l/2} g(x),$$

where

$$\int \left(\frac{1}{x_1^2+x_2^2+x_3^2} + 1\right)|g(x)|^2 dx < \infty.$$

Since

$$\frac{1+x_3^2}{x_1^2+x_2^2+x_3^2} \leqq \begin{cases} \dfrac{2}{x_1^2+x_2^2+x_3^2}, & x_1^2+x_2^2+x_3^2 < 1 \\ 2, & x_1^2+x_2^2+x_3^2 > 1, \end{cases}$$

we see from the previous estimate that

$$\int \frac{1+x_3^2}{x_2^2+x_2^2+x_3^2}|g(x)|^2 dx < \infty.$$

From this, in turn, we see that setting

$$h(x) = \left(\frac{1+x_3^2}{x_1^2+x_2^2+x_3^2}\right)^{1/2} g(x),$$

we have

$$\left(\frac{1}{x_1^2 + x_2^2 + x_3^2}\right)^{1/2} f(x) = \left(\frac{1}{1 + x_3^2}\right)^{(l+1)/2} h(x)$$

and

$$\int |h(x)|^2 dx < \infty .$$

That is to say relation (4.46) implies that $M^{(1)}\left(\frac{1}{r}\right)f$ is in $\mathfrak{B}_{l+1}^{(1)}$ and assumption $(3.11)_l$ holds. Thus we seen from Lemma 3.2 that setting

$$(4.47)^{(1)} \qquad \mathfrak{S}_k^{(1)} = \bigcap_{l=0}^{2-k} \mathfrak{D}\left(M_3\left(\frac{1}{w}\right)\right)^{l/2} \quad k = 0,1,2,$$

we have

$$(4.48)^{(1)} \qquad E\{\lambda_b\}\mathfrak{L}_2(\mathscr{E}_6) \subset \mathfrak{S}_0^{(1)}$$

and

$$(4.49)^{(1)} \qquad (\lambda_b - \text{He} + E\{\lambda_b\})^{-1}\mathfrak{S}_k^{(1)} \subset \mathfrak{S}_k^{(1)}; \qquad k = 0,1,2.$$

A similar argument, that we shall not carry out, shows that Lemma 3.2 applies to the pair of operators

$$(4.44)^{(2)} \qquad A_0^{(2)} = F^{(2)} \textit{ and } A_0^{(2)} = -2M^{(2)}\left(\frac{1}{r}\right),$$

with reference to the sets

$$(4.45)^{(2)} \qquad \mathfrak{B}_l^{(2)} = \mathfrak{D}\left(M_6\left(\frac{1}{w}\right)^{l/2}\right), \qquad l = 0,1,2.$$

Thus we see from Lemma 3.2 that setting

$$(4.47)^{(2)} \qquad \mathfrak{S}_k^{(2)} = \bigcap_{l=0}^{2-k} \mathfrak{D}\left(M_6\left(\frac{1}{w}\right)^{l/2}\right), \qquad k = 0,1,2,$$

we have

$$(4.48)^{(2)} \qquad E\{\lambda_b\}\mathfrak{L}_2(\mathscr{E}_6) \subset \mathfrak{S}_0^{(2)},$$

and

$$(4.49)^{(2)} \qquad (\lambda_b - \text{He} + E\{\lambda_b\})^{-1}\mathfrak{S}_k^{(2)} \subset \mathfrak{S}_k^{(2)}, \qquad k = 0,1,2.$$

Relations $(4.47)^{(1)}$, $(4.48)^{(1)}$, $(4.49)^{(1)}$ and $(4.47)^{(2)}$, $(4.48)^{(2)}$, $(4.49)^{(2)}$ together show that setting

$$(4.47) \qquad \mathfrak{S}_k = \mathfrak{S}^{(1)}_k \cap \mathfrak{S}_k^{(2)} \qquad k = 0,1,2$$

we have

$$E\{\lambda_b\}\mathfrak{L}_2(\mathscr{E}_6) \subset \mathfrak{S}_0, \tag{4.48}$$

and

$$(\lambda_b - \text{He} + E\{\lambda_b\})^{-1}\mathfrak{S}_k \subset \mathfrak{S}_k, \qquad k = 0,1,2. \tag{4.49}$$

In other words, at the point λ_b the operator He satisfies assumptions (3.6) and $(3.8)_k$ of Lemma 3.1 with reference to the sets of definition (4.47).

Finally we maintain that the sets of definition (4.47) satisfy assumptions $(3.7)_k$ of Lemma 3.1 with reference to the perturbation V of definition (2.7). For, suppose that f is in $\mathfrak{S}_0$, which in view of definitions (4.47), $(4.47)^{(1)}$, $(4.47)^{(2)}$ and (4.2) means that

$$\int [(1 + x_3^2)^l + (1 + x_6^2)^l]\,|f(x)|^2 dx < \infty, \qquad l = 0,1,2. \tag{4.50}$$

Since

$$(x_3 + x_6)^2 \leqq 4 \max(x_3^2, x_6^2),$$

we see that

$$(x_3 + x_6)^2[(1 + x_3^2)^l + (1 + x_6^2)^l] \leqq 8[(1 + x_3^2)^{l+1} + (1 + x_6^2)^{l+1}].$$

Insertion of this inequality in assumption (4.50) yields

$$\int (x_3 + x_6)^2[(1 + x_3^2)^l + (1 + x_6^2)^l]\,|f(x)|^2 dx < \infty, \textit{ for } l = 0,1.$$

Remembering definitions (2.7), (4.47) and (4.2) this estimate says that Vf is in $\mathfrak{S}_1$. Thus

$$V\mathfrak{S}_0 \subset \mathfrak{S}_1,$$

and we see similarly that

$$V\mathfrak{S}_k \subset \mathfrak{S}_{k+1} \textit{ for } k = 0,1,.$$

That is to say, assumptions $(3.7)_k$ hold as we have mainained.

Therefore we can conclude from Lemma 3.1 that the first two formal perturbation equations corresponding to the family $H(e)$ of definition (2.8) do admit solutions. In other words we have established the validity of condition (2.12). This completes the proof of Theorem 2.1.

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